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On a conjecture of Guillemin and Sternberg in geometric quantization of multiplicity-free symplectic spaces

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Abstract. *We study the relation between multiplicity-free symplectic manifolds and the multiplicities of group representations obtained by geometric quantization. With the help of a general equivalence theorem we can prove a conjecture of Guillemin and Sternberg in the compact Kähler case.*

0. INTRODUCTION

Quantization attempts to associate a Hilbert space X_{quantum} to a given classical phase space, a symplectic manifold (X, ω), along with a map δ from the classical observables, real functions on X, to the quantum mechanical observables, hermitian operators on X_{quantum}. The Heisenberg uncertainty principle forces δ to obey the Dirac quantization rules, i.e. δ should be an algebra homomorphism with respect to the Poisson-bracket on functions and the commutator of operators (up to a factor $\sqrt{-1}$ by physical convention).

The theory of *geometric quantization* works this out in many physical relevant cases, while fulfilling mathematical demands on rigour (see [Wo] or [Ki] for the state of the art).

An important feature of a sensible quantization is to represent a symmetry group of X unitarily on X_{quantum}. Properties of the classical action should be reflected by

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its quantized version. For example, if a compact group K acts transitively on X, there are only trivial K-invariant classical observables. By Schurs lemma the same physical property for a quantum system means that the K-representation is irreducible. Geometric quantization discovers the precise mathematical connection via the Borel-Weil theorem (see the remarks in Section 5).

If the representation of K is *multiplicity-free*, i.e. the multiplicity of every irreducible representation of K in X_{quantum} is one or zero, then K provides us with «good quantum numbers», since the algebra of bounded K-invariant operators is commutative.

«Dequantizing» this property, Guillemin and Sternberg define a K-action on a symplectic manifold to be *multiplicity-free* if the Poisson-algebra of smooth K-invariant functions $\mathcal{E}(X)^K$ is abelian. Connecting the classical and quantum mechanical notions they prove ([GS2]):

THEOREM. *(Guillemin-Sternberg) Let Q be a manifold and $K \times Q \to Q$ a smooth action of a connected compact Lie group. Then the following are equivalent:*

(1) *The action of K on T^*Q is multiplicity-free.*

(2) *$L^2_{\frac{1}{2}}(Q)$, the space of square-integrable $\frac{1}{2}$-densities on Q, is a multiplicity-free K-representation.*

Besides microlocal analysis of pseudodifferential operators, the proof uses some geometrical insights given by the map $\varphi : T^*Q \to \mathfrak{k}^*$, $\varphi(x)(\mathbf{X}) = \Theta(\mathbf{X})(x)$, where Θ is the canonical 1-form on T^*Q and $\mathbf{X}$ is in the Lie algebra $\mathfrak{k}$ of K. To generalize this, it is common to call an action of a Lie group G on a symplectic manifold *Poisson* if there exists a G-equivariant map $\varphi : X \to \mathfrak{g}^*$ such that the negative skew-gradient of $\varphi^{\mathbf{X}} = \mathbf{X} \circ \varphi$ is the fundamental vector field associated to $\mathbf{X}$ in $\mathfrak{g}$. Such a map is called a *momentum* for the G-action.

Motivated by the above theorem Guillemin and Sternberg conjecture:

Let $K \times X \to X$ be a Poisson action and $K \to U(X_{\text{quantum}})$ the representation provided by geometric quantization. Then:

$$X \text{ is multiplicity-free}$$
$$\iff X_{\text{quantum}} \text{ is a multiplicity-free } K\text{-representation}.$$

In this paper we study this conjecture with the help of two equivalence theorems, which translate the notion of a multiplicity-free space into the language of algebraic group actions on complex algebraic varieties, we proved in [Wu] (see also [HWu]). This gives a short proof of the theorem of Guillemin and Sternberg and clarifies the situation in the compact Kähler case. In fact we can prove here the following modified version of the conjecture:

THEOREM. *Let* (X,ω) *be a compact Kähler manifold,* $[\omega] \in H^2_{\mathrm{dR}}(X,\mathbb{Z})$ *and* $L = L(\omega)$ *a holomorphic line bundle with first Chern class* $[c_1(L)] = [\omega]$. *Let* $K \times X \to X$ *be a Poisson action of a connected compact group* K *of holomorphic automorphisms of* X. *Then the following are equivalent:* (1) K *acts multiplicity-free on* X. (2) $\Gamma_{\mathcal{O}}(X,L^n)$, *the space of holomorphic sections of* L^n, *is a multiplicity-free representation of* K *for all* $n \geq 0$.

We give an example to show that only assuming $X_{\text{quantum}} = \Gamma_{\mathcal{O}}(X,L)$ to be a multiplicity-free representation does *not* imply that K acts multiplicity-free on X.

In short, the organization of the paper is as follows:

Definitions and basic useful facts on *spherical* complex varieties are given in Section 1.

In Section 2 we merely state the equivalence theorem in the homogeneous case, but give the details of the proof in the compact Kähler case, since we need some fine piece of the complex analytic information found there.

Applying the equivalence theorem, we prove the theorem of Guillemin and Sternberg in Section 3, along with an easy intertwining lemma on natural representations related to homogeneous spaces.

Preparatory material on liftings of actions to bundles is the content of Section 4.

In Section 5 we study the conjecture in the compact Kähler case in detail, e.g. we derive uniqueness of the quantizing bundle in the multiplicity-free situation, and prove the above theorem.

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1. SPHERICAL VARIETIES

In non-commutative harmonic analysis of compact Lie groups the n-sphere $S^n = \mathrm{SO}(n+1,\mathbb{R})/\mathrm{SO}(n,\mathbb{R}) = K/L$ exhibits properties which are very similar to the commutative case of compact tori $(S^1)^m$; in fact the convolution algebra $C^0(L\backslash K/L)$ of L-biinvariant continuous functions on K is commutative. Since the school of Gelfand noticed the representation theoretic relevance of this property, such pairs (K,L) of a compact Lie group K and a closed subgroup L are called *spherical* or *Gelfand-pairs.*

The following characterization is well-known (see e.g. [D]):

THEOREM 1.1. *Let* K *be a connected compact Lie group and* L *a closed subgroup of* K. *Then the following are equivalent:*

(1) (K,L) *is a Gelfand-pair.*

(2) $L^2(K/L)$, *the space of measurable functions on* K *which are square integrable with respect to the Haarmeasure on* K *and almost-everywhere right-L-invariant, is a*

multiplicity-free K-representation.

(3) $\dim V^L = \dim\{v \in V \mid l \cdot v = v \; \forall l \in L\} \leq 1$ *for all irreducible complex representations of* K.

Since the set $\hat{K}$ of irreducible unitarian complex representations of K and the set $\hat{K}^{\mathbb{C}}$ of irreducible holomorphic representations of the complexification $K^{\mathbb{C}}$ of K are the same, it is not surprising that the representation $L^2(K/L)$ can be realised in a holomorphic way:

LEMMA 1.2. *Let K be a compact connected Lie group, L a closed subgroup of K and $H := L^2(K/L)$. Denote the space of regular functions on the affine algebraic variety $K^{\mathbb{C}}/L^{\mathbb{C}}$ by $V = \mathcal{O}_{\text{alg}}(K^{\mathbb{C}}/L^{\mathbb{C}})$. Then the multiplicities of an irreducible representation $\delta \in \hat{K} = \hat{K}^{\mathbb{C}}$ fulfil:*

$$m_\delta(H) = m_\delta(V) .$$

Proof. For the isotypical summand with respect to δ we have $\mathcal{O}_{\text{alg}}(K^{\mathbb{C}})_{(\delta)} \cong W \otimes W^*$ as a $K^{\mathbb{C}} \times K^{\mathbb{C}}$-module, where $W \in \delta$ and $K^{\mathbb{C}} \times K^{\mathbb{C}}$ acts by left and right translation on $K^{\mathbb{C}}$. It follows:

$$\mathcal{O}(K^{\mathbb{C}}/L^{\mathbb{C}})_{(\delta)} = \left[\mathcal{O}(K^{\mathbb{C}})_{(\delta)}\right]^{L^{\mathbb{C}}} = (W \otimes W^*)^{L^{\mathbb{C}}} = W \otimes (W^*)^{L^{\mathbb{C}}} ,$$

i.e. $m_\delta(V) = \dim(W^*)^{L^{\mathbb{C}}}$.

Analogously we have the following L^2-version of Frobenius-reciprocity for the K-action on H, which is induced by the trivial action of L on $\mathbb{C}$ (see e.g. [D]):

$$m_\delta(H) = \dim \bar{W}^L = \dim(W^*)^L .$$

By the identity principle follows $(W^*)^L = (W^*)^{L^{\mathbb{C}}}$ and thus

$$m_\delta(V) = m_\delta(H) .$$

■

Therefore a pair (K, L) is a Gelfand-pair iff $\mathcal{O}_{\text{alg}}(K^{\mathbb{C}}/L^{\mathbb{C}})$ is a multiplicity-free $K^{\mathbb{C}}$-representation.

It is not difficult to see that $\mathcal{O}_{\text{alg}}(Z)$ of an affine algebraic $K^{\mathbb{C}}$-variety is a multiplicity-free representation iff every maximal connected complex solvable subgroup of $K^{\mathbb{C}}$, shortly a Borel group in the sequel, has an open orbit in Z (see e.g. [Kr]).

This intrinsic geometric condition is fixed by the following definition:

DEFINITION. *An algebraic variety Z with an algebraic action of a reductive Lie group $K^{\mathbb{C}}$ is called spherical if every Borel group of $K^{\mathbb{C}}$ has an open orbit in Z.*

Well-known classes of examples are the torus embeddings, i.e. varieties with an action of an algebraic torus $(\mathbb{C}^*)^n$ such that there is an open orbit which is isomorphic to $(\mathbb{C}^*)^n$, and homogeneous-rational manifolds $K^{\mathbb{C}}/P$, where P contains a Borel group of $K^{\mathbb{C}}$. The latter case generalizes the flag manifolds, where $K^{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C}) = \mathrm{SU}(n, \mathbb{C})^{\mathbb{C}}$.

Homogeneous varieties $K^{\mathbb{C}}/H$, where H is an algebraic subgroup of $K^{\mathbb{C}}$, are affine iff H is reductive. Thus in the general case the regular functions on $K^{\mathbb{C}}/H$ must be replaced by sections in appropriate holomorphic bundles. We call a holomorphic line bundle $L \xrightarrow{\pi} K^{\mathbb{C}}/H$ a *homogeneous line bundle* if $K^{\mathbb{C}}$ acts holomorphically on the total space of L such that π is equivariant. In the homogeneous case the following statement is useful:

THEOREM 1.3. *For a connected complex reductive Lie group $K^{\mathbb{C}}$ and an algebraic subgroup H the following are equivalent:*

(1) *$K^{\mathbb{C}}/H$ is spherical*

(2) *The space $\Gamma_{\mathcal{O}}(K^{\mathbb{C}}/H, L)$ of holomorphic sections is a multiplicity-free $K^{\mathbb{C}}$-representation for all $K^{\mathbb{C}}$-homogeneous holomorphic line bundles L on $K^{\mathbb{C}}/H$.*

(3) *For all projective $K^{\mathbb{C}}$-varieties Z which are open $K^{\mathbb{C}}$-equivariant embeddings of $K^{\mathbb{C}}/H$, i.e. $K^{\mathbb{C}}/H \hookrightarrow Z$, the number of $K^{\mathbb{C}}$-orbits in Z is finite.*

The proof of $(1) \Leftrightarrow (2)$ is found in [KV] and of $(2) \Leftrightarrow (3)$ in [A].

2. THE EQUIVALENCE THEOREMS

To relate the notion of a spherical variety with the symplectic setting we have in mind, the following obvious definition is a useful shorthand:

DEFINITION. *Let (X, ω) be a Kähler manifold and K a connected compact Lie group which acts smoothly on X. The action is called Kähler-Poisson if the following conditions hold:*

(1) *K acts a group of biholomorphic transformations of X.*

(2) *The K-action is Poisson with respect to ω.*

We remark that a variety $K^{\mathbb{C}}/H$, where K is an algebraic subgroup of $K^{\mathbb{C}}$, can always be realized as a quasi-projective $K^{\mathbb{C}}$-orbit in a suitable complex projective space. Thus $K^{\mathbb{C}}/H$ inherits a Kähler structure such that K acts Kähler-Poisson.

In [Wu] (see also [HWu]) we prove the equivalence of the symplectic and the algebro-geometric conditions under consideration:

THEOREM 2.1. *Let K be a connected compact Lie group, $K^{\mathbb{C}}$ the complexification of K and H an algebraic subgroup of $K^{\mathbb{C}}$. Then the following conditions on $X = K^{\mathbb{C}}/H$ are equivalent:*

(1) *X is a multiplicity-free symplectic K-space with respect to a Kähler structure such that K acts Kähler-Poisson on X.*

(2) *X is a spherical $K^{\mathbb{C}}$-variety.*

REMARK. The proof involves the observation that the property multiplicity-free can be characterized by the following group theoretical condition:

$$\operatorname{codim}_X K/L = \operatorname{codim}_{T_K} T_L ,$$

where L denotes the principal isotropy subgroup of the K-action on X and the right hand of the equation is defined as the codimension of a maximal torus of L in a maximal torus of K. This is obviously independent of the chosen symplectic structure on X.

Since we need some detail facts later on we prove the analogous results in the compact Kähler case explicitely.

For the study of holomorphic automorphisms of a compact complex manifold X it is important to look at the *Albanese-map* $\psi_X : X \to \mathrm{Alb}(X)$, where $\mathrm{Alb}(X)$ is a compact complex torus and ψ_X a holomorphic map, both of them are unique by the universal property, that every holomorphic map from X to a compact complex torus factors over ψ_X (see [B] for the construction of $\mathrm{Alb}(X)$ and ψ_X). Universality implies that ψ_X is equivariant with respect to $\mathrm{Aut}_{\mathcal{O}}(X)$, the group of biholomorphic transformations of X and hence induces a morphism of Lie groups $j_X : \mathrm{Aut}_{\mathcal{O}}(X) \to \mathrm{Aut}_{\mathcal{O}}(\mathrm{Alb}(X))$ and a morphism ϱ_X of the respective Lie algebras.

In the Kähler case Fujiki proves that the subgroup $(\operatorname{kernel} j_X)^0$ of $\mathrm{Aut}_{\mathcal{O}}(X)$ with Lie algebra $\ker \varrho_X$ carries the structure of a linear algebraic group ([F]). Thus the following definition is natural:

DEFINITION. *The kernel of ϱ_X is denoted by $\mathcal{L}(X)$, the Lie algebra of linear vector fields on X.*

LEMMA 2.2. *Let X be a compact Kähler manifold and $K \times X \to X$ a Kähler-Poisson action of a compact Lie group K. Then K acts «linearly» on X, i.e. $\mathfrak{k} \to \mathcal{L}(X)$.*

Proof. We assume without loss of generality, that K acts almost effectively on X. Since K acts as a group of biholomorphic transformations, the Lie algebra $\mathfrak{k}$ is realized as a real subalgebra of $\Gamma_{\mathcal{O}}(X, TX) = \mathrm{aut}_{\mathcal{O}}(X)$. By the Poisson-assumption every fundamental vector field $\mathbf{X}$ of the K-action is the negative skew-gradient of the momentum-component $\Phi^{\mathbf{X}}(x) := \Phi(x)(\mathbf{X})$. The compactness of X implies that the function $\Phi^{\mathbf{X}}$ has extrema on X and thus $\mathbf{X}$ has zeros since $d\Phi^{\mathbf{X}}(\mathbf{Y}) = \omega(\mathbf{X},\mathbf{Y})$ for all real vector fields $\mathbf{Y}$. By the equivariance of ψ_X the induced holomorphic vector field $\varrho_X(\mathbf{X})$ has a zero on $\mathrm{Alb}(X)$ and therefore vanishes identically on the compact complex torus $\mathrm{Alb}(X)$. ∎

COROLLARY 2.3. *Let X be a compact Kähler manifold and $K \times X \to X$ an almost-effective Kähler-Poisson action of a compact Lie group K. Then the complexification $K^{\mathbb{C}}$ of K acts holomorphically and almost-effectively on X.*

Proof. Since $\mathrm{Aut}_{\mathcal{O}}(X)$ is a complex Lie group, it is enough to show:

(*) $$\mathfrak{k} \cap i\mathfrak{k} = \{0\} \qquad \text{in } \mathrm{aut}_{\mathcal{O}}(X).$$

The compact group K acts unitarily on the complex vector space $\mathfrak{k} + i\mathfrak{k}$ and therefore $\mathrm{ad} : \mathfrak{k} \cap i\mathfrak{k} \to \mathrm{End}(\mathfrak{k} \cap i\mathfrak{k})$ is trivial, i.e. $\mathfrak{k} \cap i\mathfrak{k}$ is abelian.

We define $A := \exp(\mathfrak{k} \cap i\mathfrak{k}) \subset K$ and $T := \bar{A}$, the topological closure of A in K. T is a compact torus, since the abelian group A is dense in T.

By the Borel fixed point theorem for Kähler manifolds ([So]), the abelian group A, which acts linearly on X, has a common fixed point x_0 in X. The topological closure T then also fixes x_0. We choose $\varepsilon > 0$ suitable, such that $U := \{x \in X \mid d(x, x_0) < \varepsilon\}$ is separated by the bounded holomorphic functions on U. For x in U we have $A \cdot x \subset T \cdot x \subset U$, since U is T-stable.

Since $A \cdot x$ is bounded holomorphic separable and the universal covering of $A \cdot x$, a number space $\mathbb{C}^m$, has only constant bounded holomorphic functions, A acts trivial on U and thus, by the identity principle, on X, i.e. $\mathfrak{k} \cap i\mathfrak{k} = \{0\}$. ∎

We remark that the example $K = X =$ compact complex torus shows that the corollary does not hold without the Poisson-assumption.

THEOREM 2.4. *Let X be a compact Kähler-manifold and $K \times X \to X$ a Kähler-Poisson action of a connected compact Lie group on X. Then the following are equivalent:*

(1) *X is a multiplicity-free symplectic K-space.*

(2) *X is projective and spherical with respect to the $K^{\mathbb{C}}$-action.*

Proof. (1) $\Rightarrow$ (2)

We may assume without loss of generality that K acts almost effectively. In the Kähler case the skew-complement of a subspace $V \subset T_xX$,

$$V^{\angle} := \{ w \in T_xX \mid \omega_x(v, w) = 0 \ \forall v \in V \} ,$$

and the ortho-complement

$$V^{\perp} := \{ w \in T_xX \mid w \perp v \ \forall v \in V \} ,$$

are related by $V^{\perp} = J(V^{\angle})$, where J denotes the almost complex structure on X.

Since the K-orbit of a generic point x in a multiplicity-free space X is easily seen to be coisotropic, $(T_xK \cdot x)^{\angle} \subset T_xK \cdot x$, (see [GS1] or [Wu]), the above formula for $V = T_xK \cdot x$ shows

$$\begin{aligned} T_xX &= T_xK \cdot x \oplus (T_xK \cdot x)^{\perp} = T_xK \cdot x \oplus J(T_xK \cdot x^{\angle}) \\ &= T_xK \cdot x + J(T_xK \cdot x) . \end{aligned}$$

Together with Corollary 2.3 this implies that $K^{\mathbb{C}} \cdot x$ is open in X. The equivariance of ψ_X implies on one hand that $\psi_X(K^{\mathbb{C}} \cdot x) = \mathrm{Alb}(X)$ while on the other hand $K^{\mathbb{C}}$ acts linearly by Lemma 2.2. Thus the Albanese torus reduces to a point and from $\dim_{\mathbb{C}} \mathrm{Alb}(X) = \frac{1}{2} b_1(X) = \frac{1}{2} \dim_{\mathbb{R}} H^1_{\mathrm{dR}}(X, \mathbb{R})$ it follows that the first Betti number of X must vanish.

We can now apply a theorem of E. Oeljeklaus ([O]):

Let X be an almost-homogeneous compact Kähler manifold with $b_1(X) = 0$. Then the following hold: the cohomology of the holomorphic structure sheaf vanishes: $\check{H}^i(X, \mathcal{O}) = 0$ for $i \geq 1$, $\pi_1(X) = 0$, and X is $\mathrm{Aut}_{\mathcal{O}}(X)^0$-equivariantly embedded in some projective space.

Thus we can apply Theorem 2.1 to the open orbit $K^{\mathbb{C}} \cdot x$ and it follows that $K^{\mathbb{C}} \cdot x$ and hence X itself is spherical.

(2) $\Rightarrow$ (1)

Since X is spherical, a fortiori $K^{\mathbb{C}}$ has an open dense orbit $K^{\mathbb{C}} \cdot x$ in X. By Theorem 2.1 $K^{\mathbb{C}} \cdot x$ is multiplicity-free and the denseness of $K^{\mathbb{C}} \cdot x$ implies that $\mathcal{E}(X)^K$ is also abelian. ∎

3. THE COTANGENT BUNDLE CASE

In this section we apply the equivalence theorems of Section 2 to give a short proof of the theorem of Guillemin and Sternberg.

THEOREM 3.1. (Guillemin-Sternberg) *Let K be a connected compact Lie group and L a closed subgroup of K. Consider the natural K-action on $T^*(K/L)$, which is Poisson with respect to the canonical symplectic structure. Then the following conditions are equivalent:* (1) *$T^*(K/L)$ is a multiplicity-free K-space.* (2) *$L^2(K/L)$ is a multiplicity-free K-representation.*

Proof. Following Mostow (Lemma 4.1 in [Mo2] applied to the result of Theorem 3 in [Mo1] in our situation) we identify $T^*(K/L)$ K-equivariantly with $K^{\mathbb{C}}/L^{\mathbb{C}}$.

By an affine embedding we can equip the latter with a Kähler structure such that K acts Kähler-Poisson. As we remarked after Theorem 2.1 the condition «multiplicity-free» is independent of the chosen symplectic structure on a fixed differentiable manifold.

Thus we get:

$T^*(K/L)$ is a multiplicity-free K-space
$\iff$ $K^{\mathbb{C}}/L^{\mathbb{C}}$ is a multiplicity-free K-space
$\iff$ $K^{\mathbb{C}}/L^{\mathbb{C}}$ is a spherical variety by Theorem 2.1
$\iff$ $\mathcal{O}_{\text{alg}}(K^{\mathbb{C}}/L^{\mathbb{C}})$ is a multiplicity-free $K^{\mathbb{C}}$-representation by the remarks after Lemma 1.2
$\iff$ $L^2(K/L)$ is a multiplicity-free K-representation by Lemma 1.2. ■

REMARKS. (i) If we consider an arbitrary C^∞-manifold Q, a configuration space, with a smooth K-action, then K acts in a Poisson fashion on $X = T^*Q$ as a group of point-transformations. Both of the conditions in Theorem 3.1 imply that Q is homogeneous; this is proved for (1) in [GS2] and is obvious for (2). Thus we have only to consider the above case $Q = K/L$.

(ii) In [GS2] the authors consider the Hilbert space $L^2_{\frac{1}{2}}(K/L)$ of square-integrable $\frac{1}{2}$-densities on K/L, which is the natural geometric quantization of a vertical polarized cotangent bundle (see [Wo]). For sake of completeness we show the equivalence with our representation.

LEMMA 3.2. *Let K be a connected compact Lie group and L a closed subgroup of K. Then there exists a K-equivariant intertwining isomorphism:*

$$L^2_{\frac{1}{2}}(K/L) \xrightarrow{\sim} L^2(K/L) .$$

Proof. Let μ be the unique positive measure on K/L with:

(i) μ is K-invariant

(ii) $\int_{K/L} f\, d\mu = \int_K \pi^* f\, dk$, where $\pi : K \to K/L$ denotes the canonical projection and dk the normalized Haarmeasure on K.

It is easily seen that $L^2(K/L, d\mu)$ is K-equivariant unitarily equivalent to $L^2(K/L) := L^2(K, dk)^L$. Since the measure μ is a Lebesgue-measure on K/L, i.e. in local coordinates μ is represented by a measure with a non-vanishing smooth Lebesgue-density, μ is a K-invariant trivializing section of the bundle $|\kappa|^1$ of 1-densities, which is defined by the transition functions $|f_{ij}|^1$, where f_{ij} are the transition functions of the canonical bundle $\kappa = \bigwedge^{\dim K/L} T^*(K/L)$. Since the $\frac{1}{2}$-density-bundle $|\kappa|^{\frac{1}{2}}$, defined by the transition functions $|f_{ij}|^{\frac{1}{2}} = \sqrt{|f_{ij}|}$ is also K-equivariant trivial, we find a K-invariant non-vanishing section $\sqrt{\mu}$ of $|\kappa|^{\frac{1}{2}}$ such that $\sqrt{\mu} \otimes \overline{\sqrt{\mu}} = \mu$.

Thus we get an intertwining operator

$$L^2_{\frac{1}{2}}(K/L) \to L^2(K/L, d\mu)$$

by

$$\varepsilon_{\frac{1}{2}} = f \cdot \sqrt{\mu} \mapsto f,$$

which is an K-equivariant isomorphism of Hilbert spaces. ■

4. LIFTING OF REDUCTIVE GROUP ACTIONS TO HOLOMORPHIC LINE BUNDLES

A well-known drawback of geometric quantization is the fact that a given symmetry group K of a symplectic manifold may not lift to an action on the total space of the quantizing line bundle; it can happen that only the universal covering of the group acts on the bundle space. This may destroy the compactness of the symmetry group, if K is not semi-simple. In order to assure us in the next section that in our cases this does not happen we collect here some preparatory results of a more technical nature.

Let X be a compact complex manifold and L a holomorphic line bundle on X. We define the stabilizer of the element $[L]$ in $\check{H}^1(X, \mathcal{O}^*)$, the set of equivalence classes of holomorphic line bundles on X:

$$G_{[L]} := \left\{ g \in \mathrm{Aut}_{\mathcal{O}}(X) \mid g^*L \cong L \right\}.$$

For g in $G_{[L]}$ there is a holomorphic bundle isomorphism $L \xrightarrow{\tilde{g}} g^*L$ and one has the following diagram:

$$\begin{array}{ccccc} & & \overset{\hat{g}}{\longrightarrow} & & \\ L & \underset{\tilde{g}}{\rightarrow} & g^*L & \underset{g_*}{\rightarrow} & L \\ \downarrow & & \downarrow & & \downarrow \\ X & \overset{\mathrm{Id}}{\rightarrow} & X & \overset{g}{\rightarrow} & X \\ & & \underset{g}{\longrightarrow} & & \end{array}$$

i.e. there is a bundle automorphism $\hat{g}$ of L over g. The complex Lie group $\tilde{G}_{[L]} := \{\varphi : L \to L \mid \varphi \text{ is an invertible line bundle mapping}\}$ is canonically mapped into $G_{[L]}$. This homomorphism ϱ is surjective by the above constructed $\hat{g}$ for a given g in $G_{[L]}$. The kernel of ϱ consists of the $\mathbb{C}^*$-multiplication on the line bundle and is thus central in $\tilde{G}_{[L]}$.

LEMMA 4.1. *Let $K^{\mathbb{C}}$ be a reductive subgroup of $G_{[L]}$. Then it exists a finite covering $\tilde{K} \to K$ such that $\tilde{K}^{\mathbb{C}}$ is contained in $\tilde{G}_{[L]}$ and $\varrho(\tilde{K}^{\mathbb{C}}) = K^{\mathbb{C}}$.*

REMARK. This kind of result is useful, since given an almost effective action of a compact group K on X, we can replace K by $\tilde{K}$ and we neither loose compactness nor almost-effectivity, but the new group $\tilde{K}$ acts equivariantly on the total space L and the base X.

Proof. The sequence

$$0 \to \mathbb{C}^* \to \tilde{G}_{[L]} \xrightarrow{\varrho} G_{[L]} \to 0$$

induces the sequence

$$0 \to \mathbb{C}^* \to \varrho^{-1}(K^{\mathbb{C}}) \xrightarrow{\varrho} K^{\mathbb{C}} \to 0 .$$

The homomorphism ϱ maps the radical R of $\varrho^{-1}(K^{\mathbb{C}})$, i.e. the maximal connected solvable normal subgroup of $\varrho^{-1}(K^{\mathbb{C}})$, surjective on the radical of $K^{\mathbb{C}}$, which by reductivity is isomorphic to some $(\mathbb{C}^*)^l$. As a central extension of $(\mathbb{C}^*)^l$ R is nilpotent.

Let $\tilde{R}$ denote the universal covering of R and $\tilde{Z}$ resp. Z the centers of $\tilde{R}$ resp. R. We have the following diagram of homomorphism:

$$\begin{array}{ccccc} \tilde{R} & \stackrel{\pi_1(R)}{\longrightarrow} & R & \stackrel{\mathbb{C}^*}{\longrightarrow} & (\mathbb{C}^*)^l \\ \downarrow & & \downarrow & \swarrow \tau & \\ \tilde{R}/\tilde{Z} & \stackrel{\sim}{\longleftarrow} & R/Z & & \end{array}$$

The existence of the surjective factorization homomorphism τ follows by the centrality of $\mathbb{C}^*$ in R. Since $\tilde{R}/\tilde{Z}$ is topologically a cell, τ must be trivial, i.e. R is abelian.

Provided this fact, a Levi-Malcev decomposition of $\varrho^{-1}(K^{\mathbb{C}})$ yields: $\varrho^{-1}(K^{\mathbb{C}}) = (\mathbb{C}^*)^m \cdot S$, where S is semi-simple and $(\mathbb{C}^*)^m \cap S$ is finite. We decompose further: $(\mathbb{C}^*)^m = (\ker \varrho) \cdot (\mathbb{C}^*)^l$ and set $\tilde{K} := (S^1)^l \cdot K_S$, where K_S is a maximal compact subgroup of S. It follow that $\tilde{K}^{\mathbb{C}} = (\mathbb{C}^*)^l \cdot S \xrightarrow[\varrho]{} K^{\mathbb{C}}$ is a finite covering. ∎

LEMMA 4.2. *Let X be a compact Kähler manifold and $K^{\mathbb{C}}$ a connected reductive Lie group, which acts linearly on X, i.e. $\mathfrak{k}^{\mathbb{C}} \subset \mathcal{L}(X)$ (see Section 2 for the notion of linear vector fields on Kähler manifolds). Then the action of $K^{\mathbb{C}}$ can be lifted to any holomorphic line bundle L on X.*

REMARK. By a lifting we understand here, that it exists a finite covering of $K^{\mathbb{C}}$, which acts equivariantly on L and X.

Proof. A well-known result says that the group $(\ker j_X)^0$ with Lie algebra $\mathcal{L}(X)$ acts trivial on $\check{H}^1(X, \mathcal{O}^*)$ (see e.g. [Ma]). Thus we can apply Lemma 4.1 to the reductive group $K^{\mathbb{C}} \subset (\ker j_X)^0 \subset G_{[L]}$. ∎

COROLLARY 4.3. *Let X be a compact Kähler manifold and $K \times X \to X$ a Kähler-Poisson action of a connected compact Lie group. Then the action of $K^{\mathbb{C}}$ can be lifted to any holomorphic line bundle on X.*

Proof. By Lemma 2.2 $K^{\mathbb{C}}$ acts linearly on X. ∎

5. THE COMPACT KÄHLER CASE

In this section we consider the case of compact Kähler manifolds. Application of the equivalence theorem yields a proof of a modified version (see Theorem 5.3) of the conjecture of Guillemin and Sternberg. This result is the natural analogue of the following version of the Borel-Weil theorem.

PROPOSITION 5.1. *Let (X, ω) be a compact Kähler manifold, $[\omega] \in H^2_{dR}(X, \mathbb{Z})$ and $L = L(\omega)$ a holomorphic line bundle with first Chern class $[c_1(L)] = [\omega]$. Let $K \times X \to X$ be a Kähler-Poisson action of a connected compact Lie group. Then the following conditions are equivalent:*

(1) *K acts transitively on X*

(2) *$\Gamma_{\mathcal{O}}(X, L^n)$ is an irreducible K-representation $\forall n \geq 0$.*

Proof. $(1) \Rightarrow (2)$

From the proof of Theorem 2.4 it follows a fortiori that $b_1(X) = 0$. Thus X is a homogeneous-rational manifold $K^{\mathbb{C}}/P$. Since the K-action is Kähler-Poisson, $K^{\mathbb{C}}$ acts by Corollary 4.3 on the bundle spaces L^n. Thus by the Borel-Weil-theorem (see [Se]) all the $K^{\mathbb{C}}$-representations $\Gamma_{\mathcal{O}}(X, L^n)$ are irreducible.

(2) $\Rightarrow$ (1)

Since ω is a Hodge-form, the Kodaira-embedding theorem ([Ko]) shows that for $h_0 \gg 0$, $H_{n_0} := \Gamma_{\mathcal{O}}(X, L^{n_0})$ embeds X holomorphically and $K^{\mathbb{C}}$-equivariant:

$$X \underset{\varphi_{H_{n_0}}}{\hookrightarrow} \mathbf{P}(H_{n_0}^*) .$$

Let $\pi : H_{n_0}^* \setminus \{0\} \to \mathbf{P}(H_{n_0}^*)$ denote the canonical projection. The cone $\tilde{X} := \pi^{-1}(X)$ over X is an algebraic, quasi-affine variety, acted upon by $K^{\mathbb{C}} \times \mathbb{C}^*$, where $\mathbb{C}^*$ denotes the homotheties of $H_{n_0}^*$.

Fibre coordinate development gives an injection

$$\mathcal{O}_{\mathrm{alg}}(\tilde{X}) \hookrightarrow \bigoplus_{m \geq 0} \Gamma_{\mathcal{O}}(X, L^{n_0 \cdot m}) =: \bigoplus_{m \geq 0} W_m .$$

Assuming that $\tilde{X}$ is not $(K^{\mathbb{C}} \times \mathbb{C}^*)$-homogeneous, we find a non-trivial $(K^{\mathbb{C}} \times \mathbb{C}^*)$-invariant subspace of $\bigoplus_{m \geq 0} W_m$, namely the ideal of the smallest orbit. Since $\mathbb{C}^*$ acts on W_m by the character $\lambda_m(z) = z^m$, this implies the existence of a non-trivial $K^{\mathbb{C}}$-invariant subspace of W_m for some m, contradicting condition (2). ∎

Before proving the analogous result in the multiplicity-free case, we give an uniqueness result on the «quantizing bundle».

LEMMA 5.2. *Let X, ω, L and K be as in the assumptions of Proposition 5.1 and assume furthermore that K acts multiplicity-free on X. Then the bundle L is unique up to holomorphic bundle-isomorphisms.*

Proof. We omit here the standard procedure of constructing a holomorphic line bundle L from a given integral Kähler class $[\omega]$.

By the theorem of E. Oeljeklaus quoted in the proof of Theorem 2.4, $\check{H}^i(X, \mathcal{O}) = 0$ for $i \geq 1$. Thus the long exact sequence in cohomology associated to the short exact sequence of sheaves:

$$0 \to \mathbf{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

implies that $\check{H}^1(X, \mathcal{O}^*) \xrightarrow{\cong} \check{H}^2(X, \mathbf{Z})$ by the connecting homomorphism. In other words, the topological Chern class already fixes the holomorphic equivalence class of a holomorphic line bundle. ∎

THEOREM 5.3. *For X , ω , L and K as in the assumptions of Proposition 5.1 the following conditions are equivalent:*

(1) K *acts multiplicity-free on* X

(2) $\Gamma_{\mathcal{O}}(X, L^n)$ *is a multiplicity-free* K*-representation* $\forall n \geq 0$.

Proof. (1) $\Rightarrow$ (2)

By Theorem 2.4 X is spherical. We denote the open $K^{\mathbb{C}}$-orbit in X by $K^{\mathbb{C}}/U$. Going to a finite covering, if necessary, Corollary 4.3 assures us that $K^{\mathbb{C}}$ acts on L^n. Thus the restriction $L^n|_{K^{\mathbb{C}}/U} \to K^{\mathbb{C}}/U$ is a $K^{\mathbb{C}}$-homogeneous line bundle on $K^{\mathbb{C}}/U$ and by Theorem 1.3 $\Gamma_{\mathcal{O}}(K^{\mathbb{C}}/U, L^n)$ is a multiplicity-free $K^{\mathbb{C}}$-representation.

The injectivity of the restriction of sections

$$H_n := \Gamma_{\mathcal{O}}(X, L^n) \to \Gamma_{\mathcal{O}}(K^{\mathbb{C}}/U, L^n)$$

implies that H_n is a multiplicity-free $K^{\mathbb{C}}$-module. By the identity principle this remains true for K.

(2) $\Rightarrow$ (1)

As in the proof of Prop. 5.1 we embed X in $\mathbb{P}(H^*_{n_0}) = \mathbb{P}(\Gamma_{\mathcal{O}}(X, L^{n_0})^*)$ for $h_0 \gg 0$ and consider the equivariant injection:

$$\mathcal{O}_{\text{alg}}(\tilde{X}) \hookrightarrow \bigoplus_{m \geq 0} \Gamma_{\mathcal{O}}(X, L^{n_0 \cdot m}) =: \bigoplus_{m \geq 0} W_m .$$

Since $\mathbb{C}^*$ acts on W_m by the characters $\lambda_m(z) = z^m$ condition (2) implies that $\mathcal{O}_{\text{alg}}(\tilde{X})$ is $(K^{\mathbb{C}} \times \mathbb{C}^*)$-multiplicity-free. Thus by the remarks following the proof of Lemma 1.2, every Borel subgroup of $K^{\mathbb{C}} \times \mathbb{C}^*$ has an open orbit in the affine variety $\tilde{X} \cup \{0\}$.

The equivariance of π implies that X is spherical with respect to $K^{\mathbb{C}}$. Therefore Theorem 2.4 shows that X is a multiplicity-free K-space. ∎

REMARKS. (i) By Lemma 5.2, the bundle in Proposition 5.1 and Theorem 5.3 is unique.

(ii) In the sense of geometric quantization, $H = \Gamma_{\mathcal{O}}(X, L)$ is X_{quantum} (see e.g. [Wo]). But even in the case that X is $K^{\mathbb{C}}$-almost-homogeneous, the condition that X_{quantum} is multiplicity-free does *not* imply the same for X, i.e. the conjecture of Guillemin and Sternberg is not true in the original formulation.

EXAMPLE. Let $K = SU(2, \mathbb{C})$, $X = \mathbb{P}_1(\mathbb{C}) \times \mathbb{P}_2(C)$ and $\tau : K^{\mathbb{C}} = SL(2, \mathbb{C}) \hookrightarrow SL(2, \mathbb{C}) \times SL(3, \mathbb{C}) \to \text{Aut}_{\mathcal{O}}(X)$, $\tau(A) = (A, \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix})$.

The orbit $K^{\mathbb{C}} \cdot ([1,1,0],[0,1])$ is open in X, i.e. X is almost-homogeneous. Since the dimension of a Borel group of $K^{\mathbb{C}}$ is two, X is not spherical. Theorem

2.4 or explicit computation show that K acts not multiplicity-free on X. The Fubini-Study-forms $\omega_{\mathbb{P}_n}$ on $\mathbb{P}_n(\mathbb{C})$ define via pullback a K-invariant Kähler structure $\omega := \pi_1^*\omega_{\mathbb{P}_1} + \pi_2^*\omega_{\mathbb{P}_2}$, which represents the Chern form of the bundle $L = \pi_1^* L_{\mathbb{P}_1} \otimes \pi_2^* L_{\mathbb{P}_2}$, where $L_{\mathbb{P}_n}$ denotes the hyperplane bundle on $\mathbb{P}_n(\mathbb{C})$.

The Clebesch-Gordan-formula for $\mathrm{SU}(2,\mathbb{C})$-representations:

$$V_k \otimes V_l \cong \bigoplus_{j=0}^{\min\{k,l\}} V_{k+l-2j},$$

where $V_k = \mathbb{C}_k[z_1, z_2]$, together with the fact

$$\Gamma_{\mathcal{O}}(X, L) \cong \pi_1^*\Gamma_{\mathcal{O}}(\mathbb{P}_1, L_{\mathbb{P}_1}) \otimes \pi_2^*\Gamma_{\mathcal{O}}(\mathbb{P}_2, L_{\mathbb{P}_2})$$

implies for the K-module $H = \Gamma_{\mathcal{O}}(X, L)$: $H \cong V_0 \oplus V_1 \oplus V_2$, i.e. H is a multiplicity-free representation.

(iii) Since X is compact, $\Gamma_{\mathcal{O}}(X, L^0) = \mathcal{O}(X) = \mathbb{C}$ and K acts trivial on this module. Since ω is a Kähler form, $\Gamma_{\mathcal{O}}(X, L^{-n}) = 0$ for $n > 0$. Therefore we can replace (2) by the same condition but for all $n \in \mathbb{Z} \setminus \{0\}$, indicating that quantization should take into account not only the quantizing bundle L but all powers L^n of it. This seems physically reasonable, since on the classical level multiplying ω by a constant factor changes the Hamiltonian flows only by a trivial reparametrization.

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